



Sensitivity Analysis for Linear Systems based on Reachability Sets

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2019 Conference on Decision and Control
Nice

December 11-13 2019



Outline

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Motivation

- Over-parameterized models - a complex systems can have simpler models if possible to measure their effect on the state or output.
- Multi-agent systems - there are local dynamics coupled by a network and identifying nodes with the most impact is cumbersome.
- Worst-case effect - proposed solution should look at bad trajectories.



Traditional Solution

- Traditional solutions generate signals and use the model to compute the final state or output [1].
- Solution follows a probabilistic view of the sensitivity.
- Translates a Monte Carlo approximation to the true sensitivity.

Intuition behind Reachable Sets

- Take a set describing the initial state uncertainty.
- Propagate it through time using the uncertainty set defined for all the unknown signals in the model.
- Final set represents all possible end values of the state/output.
- Iterative algorithm and Optimal for Linear Systems with no uncertainty.
- $x(0) \in X(0)$

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- $X(k) := \{p : p = Ax(k-1) + Bu(k), u(k) \in U(k), x(k-1) \in X(k-1)\}$

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- $x(k) \in X(k) / y(k) \in Y(k)$



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Problem Statement

- Find the maximum interval of each state for a given interval of the input;
- Devise methods to compute the sensitivity for:
 - One-Factor-At-Time (OAT);
 - General sensitivity.
- The algorithm should be easily extendable for more sophisticated linear systems.

Sensitivity for Linear Systems Problem

Can we compute the optimal sensitivity for linear systems with no uncertainties?

- Yes following the reachable sets defined in [2].

Reachability using Polytopes

- Linear Time-Invariant (LTI) system model:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k)$$

$$y(k) = Cx(k)$$

- Using the framework for Set-valued Observers (SVOs) [2],[3],[4],[5], we compute:

$$M(H) \begin{bmatrix} \mathbf{x} \\ \mathbf{d}(0) \\ \vdots \\ \mathbf{d}(H-1) \end{bmatrix} \leq m(H)$$

- The polytope relates all feasible instances of the state and disturbances to the system.

Sensitivity using Reachable Sets

- Sensitivity definition in a worst-case context:

$$\mathcal{S}(X_i(H), j) := x_j^{\max}(H) - x_j^{\min}(H)$$

- It is the sensitivity using the polytope produced for input i to state j .
- Given the linear description of a polytope, it corresponds to solving linear programs:

$$x_j^{\max}(H) = \max_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H)$$

$$x_j^{\min}(H) = \min_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H).$$

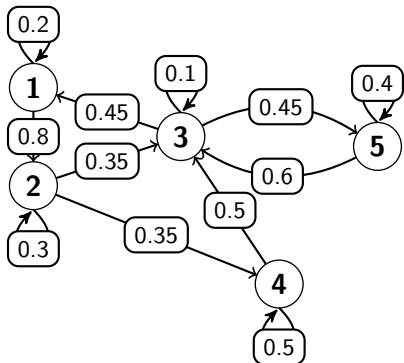
Results

- Redefining the SVO equations, it is possible to compute OAT sensitivities.
- For linear systems, the general sensitivity is equal to the sum of OAT sensitivities provided there is no initial state uncertainty.
- In the journal version, these methods are extended for the larger class of Linear Parameter-Varying (LPV) systems.
- Also, how to efficiently compute for uncertain LPV is presented where the optimality is lost.

Simulation Setup

Setup: 5 vehicles with unicycle dynamics in a formation governed by a graph.

- Vehicles start within a square of side 2 centered at the origin.
- The vehicles move using their inputs.
- After the movement, they follow a consensus algorithm.



Simulation Results

Main question: which vehicle affects the most the final state?

- Unknown initial position but known orientation;
- If one actuates node 3, it obtains the largest variation of worst-case trajectories;

# vehicle	$\mathcal{S}(X_i(H), 1)$
1	2.0467
2	2.0654
3	2.1017
4	2.0428
5	2.0763

Simulation Results

Main question: which vehicle affects the most the final state?

- With known initial position and orientation;
- The sum of OAT sensitivities are equal to the general sensitivity of 2.3329 (size of the input plus sensitivities).

# vehicle	$\mathcal{S}(X_i(H), 1)$
1	0.0467
2	0.0654
3	0.1017
4	0.0428
5	0.0763

Concluding Remarks

Contributions:

- We have presented a reachability-based method for sensitivity analysis.
- It is shown that for LTI systems, the general sensitivity is equivalent to the sum of OAT sensitivities if the initial state is known.
- In this paper, the model was assumed LTI but the SVOs work for the broader class of uncertain LPV systems (journal version).
- The more challenging case of uncertain parameters is studied in the journal version.

References

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The end

- Thank you for your time.



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