Published in *Dynamical Systems: An International Journal*
http://dx.doi.org/10.1080/14689367.2017.1288701

The Beta-Transformation's Companion Map for Pisot or Salem Numbers and their Periodic Orbits

Bruno Melo Maia

bmaia@autonoma.pt
Universidade Autónoma de Lisboa
Rua Sta. Marta 47
1150-293 Lisboa, PORTUGAL

(*Final version accepted 27 January 2017*)

The β-transformation of the unit interval is defined by $T_\beta(x) := \beta x \,(\mathrm{mod}\, 1)$. Its eventually periodic points are a subset of $[0, 1]$ intersected with the field extension $\mathbb{Q}(\beta)$.

If $\beta > 1$ is an algebraic integer of degree $d > 1$, then $\mathbb{Q}(\beta)$ is a $\mathbb{Q}$-vector space isomorphic to $\mathbb{Q}^d$, therefore the intersection of $[0, 1]$ with $\mathbb{Q}(\beta)$ is isomorphic to a domain in $\mathbb{Q}^d$. The transformation from this domain which is conjugate to the β-transformation is called the companion map, given its connection to the companion matrix of β's minimal polynomial.

The companion map and the proposed notation provide a natural setting to reformulate a classic result concerning the set of periodic points of the β-transformation for *Pisot* numbers. It also allows to visualize orbits in a d-dimensional space.

Finally, we refer connections with arithmetic codings and symbolic representations of hyperbolic toral automorphisms.

Keywords: beta-transformation, companion matrix, Pisot, Salem, periodic orbit

2010 Mathematics Subject Classification: 37A45, 37C15, 37P35, 11K16

1. Introduction

In 1957, Rényi [1] generalized integer base numbering systems to non-integer basis, introducing the β-expansion, or base β representation of real numbers. The symbols of the β-expansion of $x \in [0, 1]$ are related to iteration of x under the β-transformation $T_\beta(x) := \beta x \,(\mathrm{mod}\, 1)$. The dynamics of the β-transformation is encoded as a one sided shift on a space of symbolic sequences (for further details, please refer to [2], [3] or [4]).

The points of $[0, 1]$ having eventually periodic β-expansions are a subset of $\mathbb{Q}(\beta) \cap [0, 1]$. Bertrand and Schmidt in [5] and [6] independently proved that if β is a *Pisot* number (an algebraic integer with algebraic conjugates having modulus less than 1), then $Per(T_\beta) = \mathbb{Q}(\beta) \cap [0, 1]$.

The problem is much harder if β is a *Salem* number (an algebraic integer with algebraic conjugates of modulus less than or equal to 1, and at least one pair of them having modulus 1). The β-expansion of 1 is particularly important: whenever it is finite, we say that β is a *beta number*(which is always the case for *Pisot* numbers). Boyd proved in [7] that every *Salem* number of degree 4 is a *beta number*. In [8], Boyd produced heuristics (based on random walks) supporting the fact that every *Salem* number of degree 6 should be a *beta number*, but the same should not happen for degrees greater than 6.

In [9] Thurston observes that for the *Salem* case of degree $d \geq 4$, the orbit of 1 resembles a random walk in $\mathbb{R}^{(d-2)/2}$. It is well known that Brownian motion in dimension bigger than 2 is not recurrent. If this idea could be made rigorous, we would expect that *Salem* numbers of degree at least 8 are not *beta numbers*. This experimental observation also provides a potential explanation for Boyd's result for degree 4 *Salem* numbers, and his heuristic for degree 6 *Salem* numbers.

Section 2 sets the notation and recalls some definitions concerning β-transformations, *Pisot* and *Salem* numbers, minimal polynomials and companion matrices, which will be needed in the subsequent sections.

In Section 3 we introduce our main result. We observe that if β is an algebraic integer of degree d, then $\mathbb{Q}(\beta)$ is d-dimensional $\mathbb{Q}$-vector space with a basis $1, \beta, \ldots, \beta^{d-1}$. This basis defines a coordinate isomorphism between $\mathbb{Q}^d$ and $\mathbb{Q}(\beta)$. We define a transformation on a subset of $\mathbb{Q}^d$ which is conjugate to the restriction of T_β to $\mathbb{Q}(\beta) \cap [0,1]$. We call it the *companion map* because it is related to the companion matrix of the minimal polynomial of β. We describe the geometry of the domain of the *companion map* and establish connections with [10] and [11]. Some results in [3] were relevant for this construction.

Section 4 proposes a reformulation in our setting of the classic results from [6]. We also translate into our setting the special *Pisot* numbers of the type $\beta^2 = n\beta + 1$, referred by [6], and explain how the companion map naturally factors to the toral automorphism. As a consequence, the result in [6] concerning the period of orbits follows immediately.

2. The β-transformation and the β-expansion

For any real x, define its floor as $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}$.

Definition 2.1: *Let $\beta > 1$ and $\beta \notin \mathbb{N}$. The β-transformation is*

$$\begin{array}{rccc} T_\beta : & [0,1] & \longrightarrow & [0,1] \\ & x & \longmapsto & \beta x - \lfloor \beta x \rfloor \end{array} . \tag{1}$$

Definition 2.2: *The β-expansion of $x \in [0,1]$ is the sequence*

$$(a_1,\ a_2,\ a_3,\ \ldots)\ ,\ \text{with}\ a_n := \lfloor \beta\, T_\beta^{n-1}(x) \rfloor\ .$$

This sequence is a representation of x in basis β, because $x = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$.

Let $Per(T_\beta)$ be the set of eventually periodic points for T_β, and $\mathbb{Q}(\beta)$ be the field extension of $\mathbb{Q}$ by adjoining β. It is known that $Per(T_\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1]$ ([see 6, p. 269]). In order to study $Per(T_\beta)$, we consider the restriction of T_β to $\mathbb{Q}(\beta) \cap [0,1]$. The eventually periodic points for T_β are the numbers in $[0,1]$ having eventually periodic β-expansions.

Definition 2.3: *An* algebraic integer *is a root of a* minimal polynomial

$$p(x) = x^d + c_{d-1}\, x^{d-1} + \ldots + c_1\, x + c_0\ , \quad c_k \in \mathbb{Z}\ . \tag{2}$$

The companion matrix *of* $p(x)$ *is*

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \ddots & \vdots & -c_1 \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -c_{d-1} \end{bmatrix} . \tag{3}$$

Some authors define the *companion matrix* as the transpose of $\mathbf{C}$. The eigenvalues of $\mathbf{C}$ are the roots of $p(x)$, because its characteristic polynomial is $(-1)^d\, p(\lambda)$ (see [12, p. 339]).

Let us represent vectors in $\mathbb{R}^d$ as column matrices, and their transposes as row matrices. The row vector $\boldsymbol{\beta}^T := \begin{bmatrix} 1 & \beta & \dots & \beta^{d-1} \end{bmatrix}$ is a left eigenvector for $\mathbf{C}$ with eigenvalue β, since

$$\boldsymbol{\beta}^T \mathbf{C} = \begin{bmatrix} \beta & \beta^2 \dots & \beta^d \end{bmatrix} = \beta \boldsymbol{\beta}^T, \;\text{ given that }\; \beta^d = -\sum_{k=0}^{d-1} \beta^k c_k\ . \tag{4}$$

Definition 2.4: *A real algebraic integer $\beta > 1$ with Galois conjugates*

$$\beta_2,\ \dots,\ \beta_d$$

(a) is a Pisot *number, if $|\beta_i| < 1$ for every $2 \le i \le d$.*
(b) is a Salem *number, if $|\beta_i| \le 1$ for every $2 \le i \le d$, and at least one pair of complex conjugate roots of $p(x)$ has modulus* 1.

The *companion matrix* of $p(x)$ induces an endomorphism in the d-torus $\mathbb{R}^d/\mathbb{Z}^d$. If $|\det \mathbf{C}| = |c_0| = 1$ then $\mathbf{C} \in GL(d, \mathbb{Z})$ induces an automorphism in $\mathbb{R}^d/\mathbb{Z}^d$ (see [13, p. 46]). The minimal polynomial of a *Pisot* number induces an hyperbolic toral automorphism if and only if $|c_0| = 1$. Such *Pisot* numbers are units in the ring $\mathbb{Q}[\beta]$. Any *Salem* number necessarily induces a (non-hyperbolic) toral automorphism, because its minimal polynomial is reciprocal and has even degree $d \ge 4$, therefore $c_0 = 1$ and $\det \mathbf{C} = 1$.

3. The companion map of the $\boldsymbol{\beta}$-transformation

Assume that $\beta > 1$ is a real algebraic integer of degree $d \ge 2$, with minimal polynomial $p(x)$ and companion matrix $\mathbf{C}$. $\mathbb{Q}(\beta)$ is a $\mathbb{Q}$-vector space of dimension d and the coordinates of $\boldsymbol{\beta} := \big(1,\, \beta,\, \dots\,,\, \beta^{d-1}\big)$ form a basis (see [12, p. 429]). In this basis, the coordinate vector of $x \in \mathbb{Q}(\beta)$ is $\boldsymbol{x} := (x_0, \dots, x_{d-1}) \in \mathbb{Q}^d$, according to the isomorphism

$$\begin{array}{rccc} \phi_{\boldsymbol{\beta}}: & \mathbb{Q}^d & \longrightarrow & \mathbb{Q}(\beta) \\ & \boldsymbol{x} & \longmapsto & x = \langle\, \boldsymbol{\beta}\,,\, \boldsymbol{x}\, \rangle \end{array}, \tag{5}$$

where $\langle\, \boldsymbol{\beta}\,,\, \boldsymbol{x}\,\rangle = \sum_{k=0}^{d-1} \beta^k x_k$ is the inner product in $\mathbb{R}^d$, which can also be implemented in matrix form as $\boldsymbol{\beta}^T \boldsymbol{x}$ (or equivalently, $\boldsymbol{x}^T \boldsymbol{\beta}$).

Let $\boldsymbol{e}_1 := (1,\, 0,\, \dots,\, 0)$ be the first vector of the standard basis of $\mathbb{R}^d$.

Proposition 3.1: *Let* $R := \left\{ \boldsymbol{x} \in \mathbb{Q}^d \mid 0 \leq \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle \leq 1 \right\}$ *and* $[\mathbf{C}\boldsymbol{x}] := \lfloor \langle \boldsymbol{\beta}, \mathbf{C}\boldsymbol{x} \rangle \rfloor \boldsymbol{e}_1 = \lfloor \beta x \rfloor \boldsymbol{e}_1 \in \mathbb{Z}^d$. *The companion map of the* β*-transformation is defined by*

$$T_{\mathbf{C}} : \begin{array}{ccc} R \subset \mathbb{Q}^d & \longrightarrow & R \subset \mathbb{Q}^d \\ \boldsymbol{x} & \longmapsto & \mathbf{C}\boldsymbol{x} - [\mathbf{C}\boldsymbol{x}] \end{array} . \tag{6}$$

The companion map is conjugated by $\phi_{\boldsymbol{\beta}}$ *to the* β*-transformation, that is*

$$\begin{tikzcd} R \subset \mathbb{Q}^d \arrow[r, "T_{\mathbf{C}}"] \arrow[d, "\phi_{\boldsymbol{\beta}}"'] & R \subset \mathbb{Q}^d \arrow[d, "\phi_{\boldsymbol{\beta}}"] \\ \mathbb{Q}(\beta) \cap [0,1] \arrow[r, "T_\beta"'] & \mathbb{Q}(\beta) \cap [0,1] \end{tikzcd} .$$

Proof. By definition, $\phi_{\boldsymbol{\beta}}$ is a bijection between $R \subset \mathbb{Q}^d$ and $\mathbb{Q}(\beta) \cap [0,1]$.

Let us check that the diagram is commutative. We have seen in (4) that $\boldsymbol{\beta}^T$ is a left eigenvector of $\mathbf{C}$, with eigenvalue β. Therefore $\langle \boldsymbol{\beta}, \mathbf{C}\boldsymbol{x} \rangle = \boldsymbol{\beta}^T \mathbf{C}\boldsymbol{x} = \beta \boldsymbol{\beta}^T \boldsymbol{x} = \beta \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle$. Using this identity, we have

$$\begin{aligned} \phi_{\boldsymbol{\beta}} \circ T_{\mathbf{C}}(\boldsymbol{x}) &= \langle \boldsymbol{\beta}, \mathbf{C}\boldsymbol{x} - [\mathbf{C}\boldsymbol{x}] \rangle \\ &= \langle \boldsymbol{\beta}, \mathbf{C}\boldsymbol{x} \rangle - \langle \boldsymbol{\beta}, [\mathbf{C}\boldsymbol{x}] \rangle \\ &= \beta \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle - \lfloor \langle \boldsymbol{\beta}, \mathbf{C}\boldsymbol{x} \rangle \rfloor \\ &= \beta x - \lfloor \beta x \rfloor \\ &= T_\beta \circ \phi_{\boldsymbol{\beta}}(\boldsymbol{x}) . \end{aligned}$$

□

Remark: *Proposition 3.1 is related to a reformulation in our setting of a familiar algebraic result. Namely,* $\mathbb{Q}(\beta)$ *is isomorphic, both as* $\mathbb{Q}$*-algebras and as* $\mathbb{Q}[x]$*-modules, to* $\frac{\mathbb{Q}[x]}{<p(x)>}$. *One defines the algebra homomorphism* θ *from* $\mathbb{Q}[x]$ *to* $\mathbb{Q}(\beta)$ *by sending* x *to* β*: this is clearly surjective, since* $\mathbb{Q}(\beta)$ *is spanned as a* $\mathbb{Q}$*-vector space by the powers of* β*. The kernel of* θ *is precisely* $< p(x) >$*, given that* $p(x)$ *is the minimal polynomial of* β*. So, by the* First Isomorphism Theorem *for algebra homomorphisms,* $\frac{\mathbb{Q}[x]}{<p(x)>} \cong \mathbb{Q}(\beta)$*. This also shows that multiplication of an element of* $\frac{\mathbb{Q}[x]}{<p(x)>}$ *by* x *corresponds to multiplication of the corresponding element of* $\mathbb{Q}(\beta)$ *by* β*. Finally,* $\frac{\mathbb{Q}[x]}{<p(x)>}$ *admits a* $\mathbb{Q}$*-basis* $\{1, x, \dots, x^{d-1}\}$*, which defines an isomorphism between* $\frac{\mathbb{Q}[x]}{<p(x)>}$ *and* $\mathbb{Q}^d$*. The pair* $(\mathbb{Q}^d, \mathbf{C})$ *becomes a* $\mathbb{Q}[x]$*-module if we define* $x \cdot \mathbf{v} := \mathbf{C}\,\mathbf{v}$ *for* $\mathbf{v} \in \mathbb{Q}^d$*, hence multiplication by* β *in* $\mathbb{Q}(\beta)$ *corresponds to the linear map* $\mathbf{C}$ *in* $\mathbb{Q}^d$ *(see [14, p. 478]).*

It is known (see [13]) that $\mathbf{C}$ defines a decomposition

$$\mathbb{R}^d = E^u \oplus E^c \oplus E^s, \tag{7}$$

where E^u, E^c and E^s are the unstable, central and stable subspaces associated to the eigenvalues of $\mathbf{C}$ with modulus greater than, equal or less than one. If β is a *Pisot* number

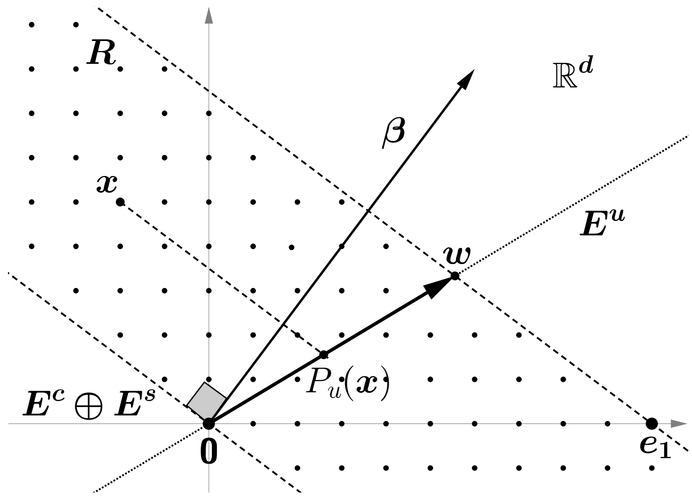

Figure 1. Domain of the Companion Map and Stable, Central and Unstable Subspaces

then $E^c = \varnothing$.

Proposition 3.2: *If* β *is a* Pisot *or a* Salem *number, then*

(a) $E^u = \ker(\mathbf{C} - \beta\mathbf{Id})$ *is a subspace of dimension* 1 *.*
(b) $E^c \oplus E^s = \ker \phi_{\boldsymbol{\beta}}$ *is a subspace of dimension* $(d-1)$ *.*

Proof. If β is a *Pisot* or a *Salem* number, then it is the only eigenvalue of $\mathbf{C}$ with modulus greater than 1, therefore the E^u is 1-dimensional and the complementary subspace $E^c \oplus E^s$ must be $(d-1)$-dimensional.

Since $\boldsymbol{\beta}^T\mathbf{C} = \beta\boldsymbol{\beta}^T \Leftrightarrow \boldsymbol{\beta}^T(\mathbf{C} - \beta\mathbf{Id}) = \mathbf{0}_{1\times d}$, then for any $\boldsymbol{x} \in \mathbb{R}^d$

$$\boldsymbol{\beta}^T(\mathbf{C} - \beta\mathbf{Id})\,\boldsymbol{x} = 0\ . \tag{8}$$

If $\boldsymbol{x} \in E^u = \ker(\mathbf{C} - \beta\mathbf{Id})$ then (8) is a tautology. But if $\boldsymbol{x} \in E^c \oplus E^s$ then (8) implies that $(\mathbf{C} - \beta\mathbf{Id})(E^c \oplus E^s) \perp \boldsymbol{\beta}$. Since $(\mathbf{C} - \beta\mathbf{Id})$ is an invertible linear map in $E^c \oplus E^s$, then $(\mathbf{C} - \beta\mathbf{Id})(E^c \oplus E^s) = E^c \oplus E^s$, therefore $E^c \oplus E^s \perp \boldsymbol{\beta} \Leftrightarrow E^c \oplus E^s = \ker \phi_{\boldsymbol{\beta}}$. □

$R \subset \mathbb{Q}^d$ is a region between $E^c \oplus E^s$ and $\boldsymbol{e}_1 + E^c \oplus E^s$, as shown in Figure 1.

Let $P_u : \mathbb{Q}^d \to E^u \subset \mathbb{R}^d$ be the projection onto the unstable space along hyperplanes parallel to $E^c \oplus E^s$. Define $\boldsymbol{w} := P_u(\boldsymbol{e}_1) = E^u \cap (\boldsymbol{e}_1 + E^c \oplus E^s)$, and

$$\begin{aligned} P_u(\boldsymbol{x}) &:= \phi_{\boldsymbol{\beta}}(\boldsymbol{x})\,\boldsymbol{w} \\ &= \langle \boldsymbol{\beta}, \boldsymbol{x}\rangle\,\boldsymbol{w}. \end{aligned} \tag{9}$$

If $\phi_{\boldsymbol{\beta}}(\boldsymbol{x}) = x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$ then $P_u(\boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}\,\boldsymbol{w} = \sum_{n=1}^{\infty} a_n \mathbf{C}^{-n}\boldsymbol{w}$. If β is a *Pisot* unit, the projection of $\boldsymbol{w}$ into the d-dimensional torus is a fundamental homoclinic point for the toral automorphism induced by $\mathbf{C}$. This connects to the theory of arithmetic codings for hyperbolic automorphisms (see [10]) and symbolic representations of expansive group automorphisms (see [11]).

4. Periodic orbits, bounded orbits and Pisot numbers

There exist d embeddings $\sigma_i : \mathbb{Q}(\beta) \to \mathbb{C}$ such that $\sigma_i(\beta) = \beta_i$ (see [12]). Let us write the forward T_β and $T_{\mathbf{C}}$ iterates as $x^{(n)} := T_\beta^n(x)$ and $\boldsymbol{x}^{(n)} := T_{\mathbf{C}}^n(\boldsymbol{x})$, and note that $x^{(n)} = \phi_{\boldsymbol{\beta}}(\boldsymbol{x}^{(n)})$. If $\boldsymbol{\beta}_i := \left(1,\, \beta_i,\, \ldots\,,\, \beta_i^{d-1}\right) \in \mathbb{C}^d$, then $\sigma_i(x^{(n)}) = \langle\, \boldsymbol{\beta}_i, \boldsymbol{x}^{(n)}\,\rangle$ and σ_1 is the inclusion map of $\mathbb{Q}(\beta)$ in $\mathbb{C}$. The crucial Lemma 2.3 of [6] can be reformulated as:

Theorem 4.1: *Let $\beta := \beta_1 > 1$ be an algebraic integer with Galois conjugates $\beta_2, \ldots, \beta_d \in \mathbb{C}$. If $x \in \mathbb{Q}(\beta) \cap [0,1]$, $\boldsymbol{x} \in R \subset \mathbb{Q}^d$ and $\phi_{\boldsymbol{\beta}}(\boldsymbol{x}) := \langle \boldsymbol{\beta}, \boldsymbol{x}\rangle = x$, then the following statements are equivalent:*

(a) $x \in Per(T_\beta)$.
(b) $\boldsymbol{x} \in Per(T_{\mathbf{C}})$.
(c) $\exists B > 0\ \forall n \in \mathbb{N}_0 \,:\, \left\|\boldsymbol{x}^{(n)}\right\| \leq B$.
(d) $\exists K > 0\ \forall 1 \leq i \leq d\ \forall n \in \mathbb{N}_0 \,:\, \left|\,\langle\, \boldsymbol{\beta}_i, \boldsymbol{x}^{(n)}\,\rangle\,\right| \leq K$.

Proof. $\phi_{\boldsymbol{\beta}}$ conjugates $(R, T_{\mathbf{C}})$ and $(\mathbb{Q}(\beta) \cap [0,1], T_\beta)$ therefore *(a)⇔(b)*.

If $\boldsymbol{x} \in Per(T_{\mathbf{C}})$ then $\left\{\boldsymbol{x}^{(n)}\right\}$ is a finite (bounded) set, thus *(b)⇒(c)*. To prove the converse, suppose *(c)* holds. Since $T_{\mathbf{C}}$ is the composition of the linear map with integer coefficients $\mathbf{C}$ with a translation in $\mathbb{Z}^d$, then $\left\{\boldsymbol{x}^{(n)}\right\}$ is a subset of the lattice $\frac{1}{q}\mathbb{Z}^d$, where q is the m.c.d. of $\boldsymbol{x}$'s coordinates. A bounded subset of a lattice is finite, therefore *(c)⇒(b)*.

If *(c)* holds, then *(d)* is true, because

$$\left|\langle\, \boldsymbol{\beta}_i, \boldsymbol{x}^{(n)}\,\rangle\right| \leq \|\boldsymbol{\beta}_i\| \cdot \left\|\boldsymbol{x}^{(n)}\right\| \leq \max_{1\leq i\leq d} \{\, \|\boldsymbol{\beta}_i\|\,\} \cdot B = K.$$

Finally, following an idea from [6, p. 272], we have

$$\begin{bmatrix} \langle\, \boldsymbol{\beta}_1, \boldsymbol{x}^{(n)}\,\rangle \\ \langle\, \boldsymbol{\beta}_2, \boldsymbol{x}^{(n)}\,\rangle \\ \vdots \\ \langle\, \boldsymbol{\beta}_d, \boldsymbol{x}^{(n)}\,\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \beta_1 & \cdots & \beta_1^{d-1} \\ 1 & \beta_2 & \cdots & \beta_2^{d-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta_d & \cdots & \beta_d^{d-1} \end{bmatrix}}_{V} \boldsymbol{x}^{(n)}\ . \tag{10}$$

If *(d)* is true, then left hand side of (10) is a bounded vector for every $n \in \mathbb{N}_0$. Since V is a non-singular *Vandermonde matrix* ($\beta_i \neq \beta_j$ for $i \neq j$), then $\boldsymbol{x}^{(n)}$ is a bounded vector (for every $n \in \mathbb{N}_0$) and we obtain *(c)*. □

The following theorem is a classic result of [6, p. 274] and [5], which we reproduce for the sake of completeness.

Theorem 4.2: *Let β be a* Pisot *number. Then $Per(T_\beta) = \mathbb{Q}(\beta) \cap [0,1]$.*

Proof. This is a reformulation in our setting of the proof in [6, p. 274].

Let $x \in \mathbb{Q}(\beta) \cap [0,1]$ and write its T_β-iterates as

$$x^{(n)} = \beta^n \left(x - \sum_{k=1}^{n} \frac{a_k}{\beta^k} \right)\ , \tag{11}$$

where $(a_1, a_2, \ldots)$ is the β-expansion of x and $0 \leq a_k \leq \lfloor\beta\rfloor$.

Applying σ_i to both members of this equation, we obtain

$$\langle \boldsymbol{\beta}_i, \boldsymbol{x}^{(n)} \rangle = \beta_i^n \left(\langle \boldsymbol{\beta}_i, \boldsymbol{x} \rangle - \sum_{k=1}^{n} \frac{a_k}{\beta_i^k} \right) . \tag{12}$$

By definition $\beta_1 := \beta$, and it follows that $|\langle \boldsymbol{\beta}_1, \boldsymbol{x}^{(n)} \rangle| = |x^{(n)}| \leq 1$.
If $2 \leq i \leq d$ then $|\beta_i| < 1$, because β is a *Pisot* number, therefore

$$\begin{aligned} |\langle \boldsymbol{\beta}_i, \boldsymbol{x}^{(n)} \rangle| &\leq |\beta_i|^n \cdot |\langle \boldsymbol{\beta}_i, \boldsymbol{x} \rangle| + \lfloor \beta \rfloor \cdot \sum_{k=1}^{n} |\beta_i|^{n-k} \\ &\leq \max_i |\langle \boldsymbol{\beta}_i, \boldsymbol{x} \rangle| + \lfloor \beta \rfloor \cdot (1 - |\beta_i|)^{-1} \leq K. \end{aligned} \tag{13}$$

This proves *(d)* of Theorem 4.1, which is equivalent to $x \in Per(T_\beta)$. □

The dynamical systems $(R, T_{\mathbf{C}})$ and $(\mathbb{Q}^d/\mathbb{Z}^d, \overline{C})$, where $\overline{C}$ is the toral endo/automorphism induced by $\mathbf{C}$, are semi-conjugated by

$$\begin{array}{cccc} \pi : & \mathbb{Q}^d & \longrightarrow & \mathbb{Q}^d/\mathbb{Z}^d \\ & \boldsymbol{x} & \longmapsto & \boldsymbol{x} + \mathbb{Z}^d \end{array} , \tag{14}$$

since π is a continuous surjection and the following diagram commutes

$$\begin{tikzcd} R \subset \mathbb{Q}^d \arrow[r, "T_{\mathbf{C}}"] \arrow[d, "\pi"'] & R \subset \mathbb{Q}^d \arrow[d, "\pi"] \\ \mathbb{Q}^d/\mathbb{Z}^d \arrow[r, "\overline{C}"'] & \mathbb{Q}^d/\mathbb{Z}^d \end{tikzcd}$$

$$\pi \circ T_{\mathbf{C}}(\boldsymbol{x}) = \pi\left(\mathbf{C}\boldsymbol{x} - [\mathbf{C}\boldsymbol{x}]\right) = \mathbf{C}\boldsymbol{x} + \mathbb{Z}^d = \overline{C} \circ \pi(\boldsymbol{x}) . \tag{15}$$

The period of $\boldsymbol{x}$ for $T_{\mathbf{C}}$ is a multiple of the period of $\pi(\boldsymbol{x})$ for $\overline{C}$. This is the reformulation in our setting of Proposition 3.3. from [6, p. 275].

Theorem 3.4. of [6, p. 275] proves that if β is a *Pisot* number with minimal polynomial $p(x) = x^2 - nx - 1$, for some $n \in \mathbb{N}$, then every $x \in \mathbb{Q}(\beta) \cap [0, 1]$ has strictly periodic β-expansion. It is also claimed that the period of x for T_β is the same as the period of $\pi(\boldsymbol{x})$ for $\overline{C}$. We note that this is the only case where the companion matrix is symmetric

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix} , \tag{16}$$

hence $\mathbb{R}^2 = E^u \oplus E^s$ is an orthogonal decomposition. It would be interesting to pursue the study of this particular case, and find out what distinguishes it from the general *Pisot* cases.

5. Conclusion

When β is a *Pisot* or a *Salem* number of degree $d > 1$, the β-transformation is conjugate to a transformation on a subset of $\mathbb{Q}^d$. We call it the companion map for the β-transformation, because its definition is related to the companion matrix of the minimal polynomial of β. This setting emphasizes the Dynamical Systems' point of view, rather than the original number theoretical framework.

The companion map and the associated notation provide a good framework to reformulate the classical result from [6] concerning the set of eventually periodic points of the β-transformation when β is a *Pisot* number. Furthermore, it gives a geometric representation of the orbits of the β-transformation in a d-dimensional space, which could yield insight useful for further study of this area.

Acknowledgement(s)

I would like to thank Anthony Manning for his suggestions concerning this work over a long period of time. I'm also thankful to Pedro Miguel Duarte for his encouragement and feedback concerning the preliminary drafts. I would like to thank Kenneth A. Brown for his suggestions concerning the remark to Proposition 3.1.
Last but not the least, I would like to thank my good friend Daniel Thompson, whose knowledge and support was invaluable towards the completion of this work.

References

[1] Renyi A. Representations of real numbers and their ergodic properties. Acta Math Hungar. 1957;8:477–493.
[2] Parry W. On the β-expansions of real numbers. Acta Math Hungar. 1960;11:401–416.
[3] Maia B. An equivalent system for studying periodic points of the beta transformation for a Pisot or a Salem number [dissertation]. Warwick; 2008.
[4] Climenhaga V, Thompson D. Intrinsic ergodicity beyond specification: β-shifts, s-gap shifts, and their factors. Isr J Math. 2012;192, Part B:785–817.
[5] Bertrand A. Développements en base de Pisot et répartition modulo 1. C R Acad Sci Paris. 1977;Sér. A-B, 285 (6):419–421.
[6] Schmidt K. On periodic expansions of Pisot numbers and Salem numbers. Bull London Math Soc. 1980;12:269–278.
[7] Boyd D. Salem numbers of degree four have periodic expansions. In: de Gruyter BNY, editor. Théorie des nombres (Conference Proceedings, Banff); 1988. p. 57–64.
[8] Boyd D. On the beta expansion for Salem numbers of degree 6. Math Comp. 1996;65:861–875.
[9] Thurston W. Entropy in dimension one. 2014.
[10] Sidorov N, Vershik A. Bijective arithmetic codings of the 2-torus, and binary quadratic forms. J Dynam Cont Sys. 1998;4:365–400.
[11] Lindenstrauss E, Schmidt K. Symbolic representations of nonexpansive group automorphisms. Isr J Math. 2005;149:227–266.
[12] Birkhoff G, MacLane S. A survey of modern Algebra. Macmillan Publishing Co.; 1977.
[13] Katok A, Hasselblatt B. Introduction to the modern theory of dynamical systems. Cambridge University Press; 1995.
[14] Artin M. Algebra. Prentice Hall; 1991.