

Polchinski ERG Equation and 2D Scalar Field Theory ^a

Yuri Kubyshin ^b, Rui Neves and Robertus Potting
 Department of Physics, Universidade do Algarve, 8000 Faro, Portugal

Abstract

We investigate a Z_2 -symmetric scalar field theory in two dimensions using the Polchinski exact renormalization group equation expanded to second order in the derivative expansion. We find preliminary evidence that the Polchinski equation is able to describe the non-perturbative infinite set of fixed points in the theory space, corresponding to the minimal unitary series of 2D conformal field theories. We compute the anomalous scaling dimension η and the correlation length critical exponent ν showing that an accurate fit to conformal field theory selects particular regulating functions.

1 Introduction

Since its very origins, the exact renormalization group (ERG)^{1,2,3,4} has proved to be a powerful tool for studies of non-perturbative effects in quantum field theory (see recent reviews in^{5,6}).

A particularly interesting case is that of an effective scalar field theory in two dimensions. As it was first conjectured by Zamolodchikov⁷, for a Z_2 symmetric theory there should exist an infinite set of non-perturbative fixed points corresponding to the unitary minimal series of $(p, p+1)$ conformal field theories, where $p = 3, 4, \dots, \infty$ ⁸. Morris⁹ showed numerically that such points do exist. The calculation was performed with a reparametrization invariant version of the Legendre ERG equation¹⁰ expanded in powers of derivatives. It was also pointed out there that to the level of the local potential approximation only the continuum limits described by periodic solutions and corresponding to critical sine-Gordon models could be obtained. To find the expected set of fixed points the calculations had to be taken to the next order in the derivative expansion. This constituted a manifestation of the non-perturbative nature of the phenomena, and remarkably the Legendre ERG equation was powerful enough to locate and describe with good accuracy the expected set of 2D field theories.

In this work we study the same Z_2 symmetric scalar field theories in two dimensions but now with the Polchinski ERG equation⁴. We present prelim-

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^bOn leave of absence from the Institute for Nuclear Physics, Moscow State University, 117899 Moscow, Russia.

inary results which complement the results obtained with the Legendre ERG equation. In Sect. 2 we follow the article by Ball et al.¹¹ to present the basic equations of the formalism. This will allow us to set up notation for Sect. 3 where we analyse the equations to second order in the derivative expansion. In Sect. 4 we discuss the results and present our conclusions.

2 Polchinski equation and derivative expansion

The Polchinski equation⁴ for a scalar theory can be written as follows¹²

$$\begin{aligned} \frac{\partial}{\partial t} \hat{S} &= \int_{\hat{p}} K'(\hat{p}^2) \left[\frac{\delta \hat{S}}{\delta \hat{\varphi}_p} \frac{\delta \hat{S}}{\delta \hat{\varphi}_{-p}} - \frac{\delta^2 \hat{S}}{\delta \hat{\varphi}_p \delta \hat{\varphi}_{-p}} \right] + dS + \\ &\int_{\hat{p}} \left[1 - \frac{d}{2} - \frac{\eta(t)}{2} - 2\hat{p}^2 \frac{K'(\hat{p}^2)}{K(\hat{p}^2)} \right] \hat{\varphi}_p \frac{\delta \hat{S}}{\delta \hat{\varphi}_p} - \int_{\hat{p}} \hat{\varphi}_p \hat{p} \cdot \frac{\partial'}{\partial \hat{p}} \frac{\delta \hat{S}}{\delta \hat{\varphi}_p}. \end{aligned} \quad (1)$$

Here \hat{S} is a general Wilsonian action which can be written in terms of dimensionless variables as follows

$$\hat{S}[\hat{\varphi}; t] = \frac{1}{2} \int_p \hat{\varphi}_p p^2 (K(\hat{p}^2))^{-1} \hat{\varphi}_{-p} + \hat{S}_{int}[\hat{\varphi}; t], \quad (2)$$

$$\hat{S}_{int}[\hat{\varphi}; t] = \int dy \left[v(\hat{\phi}(y), t) + z(\hat{\phi}(y), t) \left(\frac{\partial \hat{\phi}}{\partial y^\mu} \right)^2 + \dots \right]. \quad (3)$$

In Eq. (1) the partial derivative on \hat{S} means it only acts on the explicit $t = \ln(\Lambda_0/\Lambda)$ dependence of the couplings and the prime in the momentum derivative means it does not act on the delta function of the energy-momentum conservation, and $\int_{\hat{p}} \equiv \int \frac{d^d \hat{p}}{(2\pi)^d}$. $K(\hat{p}^2)$ is a (smooth) regulating function which damps the high energy modes satisfying the normalization condition $K(0) = 1$. The renormalized field $\hat{\varphi}_p$ changes with scale according to

$$\Lambda \frac{\partial}{\partial \Lambda} \hat{\varphi}_p = \left[1 + \frac{d}{2} - \frac{1}{2} \eta(t) \right] \hat{\varphi}_p, \quad (4)$$

where $\eta(t)$ is the anomalous scaling dimension.

To the second order in the derivative expansion we consider the two terms which are written explicitly in Eq. (3). Within this approximation the Polchinski ERG equation reduces to the following system¹¹

$$\dot{f} = f'' + 2Az' - 2ff' + \Delta^+ f + \Delta^- x f', \quad (5)$$

$$\dot{z} = z'' + Bf'^2 - 4zf' - 2z'f + \Delta^- x z' - \eta z - \eta/2, \quad (6)$$

where $\Delta^\pm = 1 \pm d/2 - \eta/2$, $f(x) = v'(x)$, $x \equiv \hat{\varphi}$ and the potentials $v(x, t)$ and $z(x, t)$ are defined in Eq. (3). The dots and primes denote the partial derivatives with respect to t and x respectively. The parameters A and B reflect the scheme dependence of the equations and are equal to $A = (I_1 K_0)/I_0$, $B = K_1/K_0^2$. Here K_n and I_n , $n = 0, 1, \dots$ parametrize the regulating function in Eq. (2) and are defined by

$$\begin{aligned} K_n &= (-1)^{n+1} K^{(n+1)}(0), \\ I_n &= - \int_{\hat{p}} (\hat{p}^2)^n K'(\hat{p}^2) = -\Omega_d \int_0^\infty dz z^{d/2-1-n} K'(z), \end{aligned}$$

where $K^{(n)}$ stands for the n -th derivative of K and $\Omega_d = 2/(\Gamma(d/2)(4\pi)^d)$.

In the next section we search for fixed-point solutions, i.e. for functions $f(x)$ and $z(x)$ which are independent of t and satisfy the system

$$\begin{aligned} f'' + 2Az' - 2ff' + \Delta^+ f + \Delta^- x f' &= 0, \\ z'' + Bf'^2 - 4zf' - 2z'f + \Delta^- x z' - \eta z - \eta/2 &= 0. \end{aligned} \tag{7}$$

We will choose the initial conditions (according to the terminology adopted in the literature on the ERG equations) set by the Z_2 symmetry: $f(0) = 0$ and $z'(0) = 0$ and by the normalization condition: $z(0) = 0$. For the value of the first derivative of $f(x)$ at the origin we will take the condition $f'(0) = \gamma$, where γ is a free parameter. The anomalous dimension $\eta(t)$ at a fixed point becomes the critical exponent η_* .

3 Fixed points and critical exponents

To solve Eqs. (7), (8) for $d = 2$ we consider the recursive numerical method already tested for $d = 3$ ¹¹. The physical fixed point solutions $f_*(x)$, $z_*(x)$ at the fixed point value $\eta = \eta_*$ are regular for $x > 0$ and have a certain asymptotic behavior as $x \rightarrow +\infty$. Thus the natural method for finding the correct numerical solution is to select those which can be integrated as far as possible in x . A generic solution will end at a sharp singularity for a finite value of x . The difficulty lies in the nonlinear and stiff nature of the equations and the need to fine tune η and γ . This makes the direct integration of the system too hard. One way out is to solve it recursively.

Unlike the case $d = 3$ studied in a number of articles^{11,13}, one faces an additional difficulty in two dimensions. It is not possible to start the iterative procedure by setting in Eq. (7) $z = 0$ and $\eta = 0$ as it is prescribed by the consistency of the leading approximation. For $d = 2$ the Polchinski equation in the leading order has only periodic or singular solutions for all values of γ .

To overcome this difficulty one has to consider $\eta \neq 0$ as the initial value to start the iterations. Consequently, an analysis of the leading order Polchinski equation

$$f'' - 2ff' + \Delta^+ f + \Delta^- xf' = 0, \quad (9)$$

with the initial conditions $f(0) = 0$, $f'(0) = \gamma$ for non-zero η is required.

We studied Eq. (9) for $d = 2$ numerically for a wide range of values of η and γ . Our results show that for each $0 < \eta \leq 1$ we can fine tune γ in such a way as to obtain a non-trivial regular fixed point solution. The set of such values (η, γ) form a discrete series of continuous lines $\eta(\gamma)$ (see Fig. 1). In fact simple arguments can be presented which explain the appearance of the lines in the parameter space corresponding to regular fixed-point solutions $f_*(x)$. Let $x_0(\eta, \gamma)$ denote position of the pole of a generic solution of Eq. (9). Suppose that for some values (η', γ') the solution is regular, i.e. $x_0(\eta', \gamma') = +\infty$. Let us take another value η'' sufficiently close to η' . Assuming that the function $x_0(\eta, \gamma)$ is continuous, it is clear that there should exist the value γ'' such that again $x_0(\eta'', \gamma'') = +\infty$. Hence there is a line of the "constant value" $x_0(\eta, \gamma) = +\infty$ in the parameter space. When we move along a fixed line the solutions $f_*(x)$ do not change their shape significantly. Moreover, their shape follows a regular pattern when passing from one curve to the other similar to solutions obtained by Morris⁹. This can be considered as a sign for the existence of the infinite discrete set of fixed points corresponding to the minimal unitary series of conformal models. We also would like to note that for $-2 < \eta < 6$ and $-1 < \gamma < 1$ there are no other non-trivial fixed-point solutions besides the ones corresponding to the lines discussed here.

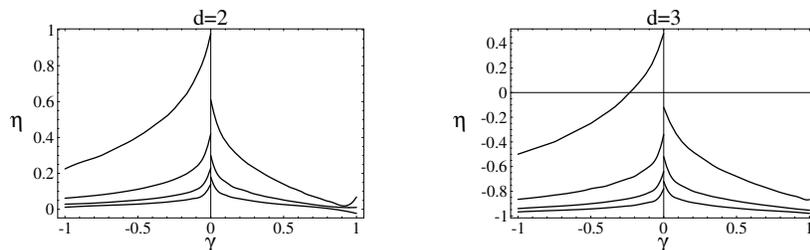


Figure 1: Non-trivial fixed-point lines for $d = 2$ and $d = 3$. For clarity, only the first 7 lines are shown.

We would like to note that a similar picture takes place in other dimensions. For $d = 3$ we found the same discrete set of lines in the (γ, η) -plane corresponding to regular solutions of Eq. (9), but in this case they are situated in the interval $-1 < \eta \leq 1/2$ (see Fig. 1). As one can see there is a line (upper

line in Fig. 1) which crosses the γ -axis at $\gamma_* = -0.229\dots$. This is the value of the parameter γ for which a non-trivial fixed point solution of the Polchinski equation was found in the leading order (local potential approximation)¹¹. The important observation is that there is only one line in the (γ, η) -plane with positive values of η . Since according to general arguments at physical fixed points $\eta_* > 0$, this suggests that for $d = 3$ there is only one non-trivial fixed point. This is the Wilson-Fischer fixed point found in numerous previous studies¹³.

Recall that for $d = 2$ all the lines are situated in the $\eta > 0$ half-plane, hence one can expect an infinite number of non-trivial fixed points. One more remark is relevant here. By a simple scaling analysis of Eq. (9) it can be shown that there is a certain mapping between the lines $\eta(\gamma)$ in different dimensions. When we pass from one dimension to another the line experiences a vertical shift and scaling transformation. More details about this mapping will be presented elsewhere.

We now pass to the study of the system (7), (8). We have seen that there are families of solutions of Eq. (9) corresponding to a given fixed point. It turns out that when the second equation of the system is taken into account, this degeneracy disappears. We solved the system using the following iteration procedure developed by Ball et al.¹¹. First we set $z(x) = 0$, choose some initial value $\eta = \eta_0$ and fine tune γ to the value $\gamma = \gamma_0$ corresponding to the regular solution $f_0(x)$ of the first equation (7) (or (9)). Of course, the point (η_0, γ_0) lies on one of the lines described above (see Fig. 1)). As the next step we insert the function $f_0(x)$ into the second equation (8) for a fixed B and fine tune η to the value $\eta = \eta_1$ for which a regular solution $z_1(x)$ exists. Then we substitute $z_1(x)$ and η_1 into the first equation of the system and find a regular solution for a fixed value of A thus obtaining a new value $\gamma = \gamma_1$ and a new function $f_1(x)$. We repeat this process keeping A and B fixed. As a result a sequence of functions $f_0(x), z_1(x), f_1(x), z_2(x), \dots$, and a sequence of numbers $\gamma_0, \eta_1, \gamma_1, \eta_2, \dots$, are obtained, and we test them for convergence.

For $d = 3$ and for the relevant line, associated with the Wilson-Fischer fixed point, we confirmed the results by Ball et al.¹¹. The new feature in our calculations is that we took $\eta_0 \neq 0$ as the initial value of the iterating procedure, whereas in¹¹ only $\eta_0 = 0$ was considered. We conclude that the numerical method converges and that the rate of convergence is controlled by A for fixed η_0 . The best A was shown to correspond to the inflexion point where $\Delta\eta = \eta_2 - \eta_1$ changes sign. The important observation is that the final values η_* and γ_* to which the iterations converge (i.e. the fixed-point values) do not depend on the initial value η_0 . When η_0 is closer to the fixed-point value the rate of convergence is of course faster. The final value η_* depends

on B linearly. For $A = 0$ the two equations decouple and there is no need for iterations to find a solution of the system. We just have to adjust B such that $\eta_1 = \eta_0$. For the Wilson-Fischer fixed point $\eta_0 = \eta_1 = 0.04$ we found $B = 0.666768\dots$ and $\gamma_* = \gamma_0 = -0.197435\dots$

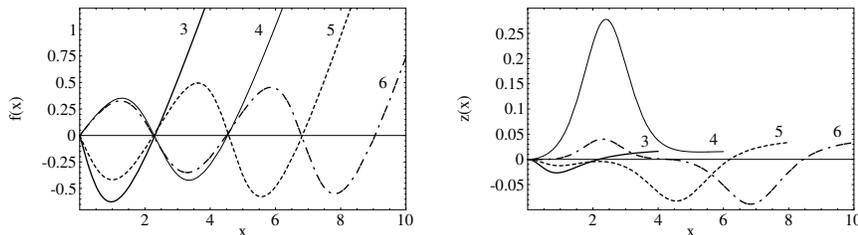


Figure 2: The solutions $f(x)$ and $z(x)$ for $p = 3, 4, 5, 6$.

For $d = 2$ the situation is totally different. For any line $\eta(\gamma)$ we start with and the initial value (η_0, γ_0) the iterative procedure turns out to be divergent if $A \neq 0$. Only for $A = 0$ we have been able to find a solution to the now decoupled system (7), (8) by adjusting B . Similar to the case $d = 3$ we have not found any natural criteria to select the value of B since for each line B depends monotonically on η decreasing as $\eta \rightarrow 0$. To determine the fixed-point solutions, corresponding to the minimal unitary series of conformal field theories, we have fixed the value of B by a fit to the series of exact values for the anomalous scaling dimension η_* . In this way we found $B \approx 0.25$. The fixed-point solutions for $f_*(x)$ display regular behaviour and are reminiscent of those obtained by Morris⁹. In particular they have $p - 2$ extrema, $p = 3, 4, \dots$ (see Fig. 2). The fixed point solutions for $z_*(x)$ have the same pattern of extrema, though their profiles are different from those of Morris.

Next for $B = 1/4$ and corresponding values $\eta = \eta_*$ we have calculated the critical exponent ν . For this we considered perturbations of the functions $f(x)$ and $z(x)$ around the fixed-point solutions,

$$f(x) = f_*(x) + \sum_{n=1}^{\infty} g_n(x) e^{\lambda_n t}, \quad z(x) = z_*(x) + \sum_{n=1}^{\infty} h_n(x) e^{\lambda_n t},$$

and substituted them into Eqs. (5), (6). After linearization we obtained the system

$$\begin{aligned} g_n'' - 2g_n f_*' - 2f_* g_n' + \Delta_*^+ g_n + \Delta_*^- x g_n' &= \lambda_n g_n, \\ h_n'' - 2f_* h_n' - 2g_n z_*' - 4f_*' h_n - 4g_n' z_* \\ &+ \Delta_*^- x h_n' - \eta_* h_n + 2B f_*' g_n' &= \lambda_n h_n, \end{aligned}$$

where Δ_*^+ and Δ_*^- are calculated for $\eta = \eta_*$.

For Z_2 perturbations the initial conditions are $g_n(0) = 0$, $h_n'(0) = 0$. We also imposed the normalization condition $g_n'(0) = 1$. Away from the fixed point (but sufficiently close to it) we relaxed h_n to be different from zero, $h_n = \delta$. Then λ_n and δ were fine tuned so that polynomially growing eigenfunctions were obtained. For $d = 2$, $A = 0$ and $B = 1/4$ we have calculated the critical exponent $\nu = 1/\lambda_1$. The results for η_* and ν are given in Table 1. We conclude that the Polchinski ERG equation gives the values for the critical exponent which match quite well the exact conformal field theory values⁷ and the results by Morris⁹, also included in the table.

We note that the best fit value for B that we have found actually corresponds to a well defined subset of the regulating functions. An explicit example is $K(x) = (1+ax+bx^2)e^{-x}$, where $a+2b+1 = 0$ to ensure $I_1 = 0$ and $I_0 = \Omega_d$, and $a = -5 \pm 2\sqrt{6}$ to give $B = 1/4$.

p	3	4	5	6	7	8	9	10	11	12
η_*	.204	.183	.0865	.0708	.0496	.0413	.0328	.0281	.0234	.0206
η_{CFT}	.25	.15	.10	.0714	.0536	.0417	.0333	.0273	.0227	.0192
η'	.309	.200	.131	.0920	.0679	.0521	.0412	.0334	.0277	.0233
ν	1.41	.560	.527	.521	.515	.512	.509	.508	.507	.506
ν_{CFT}	1	.556	.536	.525	.519	.514	.511	.509	.508	.506
ν'	.863	.566	.545	.531	.523	.517	.514	.511	.509	.508

Table 1: The critical exponents η_* and ν obtained, as compared to the values obtained by Morris⁹ (η' , ν') and the exact results of conformal field theory (η_{CFT} , ν_{CFT}).

4 Discussion and conclusions

In this work we have studied the solutions of the Polchinski ERG equation for an effective Z_2 -symmetric scalar field theory in the two-dimensional space R^2 . We have seen that this equation provides a reliable non-perturbative evidence for the existence of the fixed-point solutions corresponding to the minimal unitary series of conformal field theories and allows to calculate the anomalous dimension and the critical exponents with good accuracy. This constitutes another positive test of the power of the ERG approach. At the same time our studies are complementary to similar calculations within the ERG approach based on the equation for the Legendre action⁹.

As mentioned above, in the leading order of the derivative expansion (local potential approximation) the consistent value for η is 0 and only periodic sine-

Gordon type fixed-point solutions can be obtained. However, we have found that there are continuous families of fixed-point solutions corresponding to a series of lines $\eta(\gamma)$ in the (γ, η) plane which have a part with positive values $\eta > 0$. It was also argued that these lines correspond to the multicritical fixed points of the theory. It is by taking into account the second order in the derivative expansion that we find isolated fixed-point solutions. We have found the first 10 points out of the infinite series and calculated the critical exponent ν for them. The results depend on the choice of the regulating function. The value of B , for which a regular solution exists, depends linearly on η , so the criterium of minimal sensitivity cannot be applied to fix B . The best fit to the conformal field theory values of η gives us $A = 0, B \approx 0.25$. No other regulating functions have been found to work. For $d = 2$ whenever $A \neq 0$, the iterative procedure is not seen to converge. In fact, for $A = 0$ there is no need for iterations since the two equations of the Polchinski approach decouple. This is in sharp contrast to the case of $d = 3$ where convergence was checked for $A \neq 0$ ¹¹. For the values $A = 0, B = 1/4$, giving the best fit, our results are comparable in accuracy and sometimes better (the accuracy also increases with multicriticality) than those of Morris⁹. We would like to note that fixing B by the best fit to exact results for the anomalous dimensions η is reminiscent of fixing the renormalization scheme dependence in the perturbative renormalization group. It is also similar to fixing the regulator by the condition of the reparametrization invariance for the ERG equation for the Legendre action (that corresponds to the limiting case of $A = 0, B = \infty$)⁹.

Another important point we would like to mention is that in our analysis we have not found non-trivial fixed points other than those corresponding to the minimal models. This is what one expects from Zamolodchikov's c -theorem¹⁴. The conclusion is already clear from the analysis of the leading order Polchinski equation for $\eta \neq 0$ when the lines in the (γ, η) -plane corresponding to non-trivial fixed points are plotted.

As final comments we would like to mention that to obtain an estimate of the error of our numerical results in Table 1 one needs to carry out the calculations to the next order of the derivative expansion. It would be also interesting to expand the analysis for higher dimension operators.

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