Sensitivity Analysis for Linear Systems based on Reachability Sets

D. Silvestre, P. Rosa, J. Hespanha and C. Silvestre

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Outline

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Motivation

- Over-parameterized models - a complex systems can have simpler models if possible to measure their effect on the state or output.
- Multi-agent systems - there are local dynamics coupled by a network and identifying nodes with the most impact is cumbersome.
- Worst-case effect - proposed solution should look at bad trajectories.

Traditional Solution

- Traditional solutions generate signals and use the model to compute the final state or output [1].
- Solution follows a probabilistic view of the sensitivity.
- Translates a Monte Carlo approximation to the true sensitivity.

Intuition behind Reachable Sets

- Take a set describing the initial state uncertainty.
- Propagate it through time using the uncertainty set defined for all the unknown signals in the model.
- Final set represents all possible end values of the state/output.
- Iterative algorithm and Optimal for Linear Systems with no uncertainty.
- $x(0) \in X(0)$

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- Iterative algorithm and Optimal for Linear Systems with no uncertainty.
- $X(k) := \{p : p = Ax(k-1) + Bu(k), u(k) \in U(k), x(k-1) \in X(k-1)\}$

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- $x(k) \in X(k)$ / $y(k) \in Y(k)$

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Problem Statement

- Find the maximum interval of each state for a given interval of the input;
- Devise methods to compute the sensitivity for:
  - One-Factor-At-Time (OAT);
  - General sensitivity.
- The algorithm should be easily extendable for more sophisticated linear systems.

Sensitivity for Linear Systems Problem

Can we compute the optimal sensitivity for linear systems with no uncertainties?

- Yes following the reachable sets defined in [2].

Reachability using Polytopes

- Linear Time-Invariant (LTI) system model:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + Ed(k) \\
y(k) &= Cx(k)
\end{aligned}
$$

- Using the framework for Set-valued Observers (SVOs) [2],[3],[4],[5], we compute:

$$
M(H) \begin{bmatrix} \mathbf{x} \\ \mathbf{d(0)} \\ \vdots \\ \mathbf{d(H-1)} \end{bmatrix} \leq m(H)
$$

- The polytope relates all feasible instances of the state and disturbances to the system.

Sensitivity using Reachable Sets

- Sensitivity definition in a worst-case context:

$$\mathcal{S}(X_i(H), j) := x_j^{\max}(H) - x_j^{\min}(H)$$

- It is the sensitivity using the polytope produced for input i to state j.
- Given the linear description of a polytope, it corresponds to solving linear programs:

$$x_j^{\max}(H) = \max_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H)$$

$$x_j^{\min}(H) = \min_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H).$$

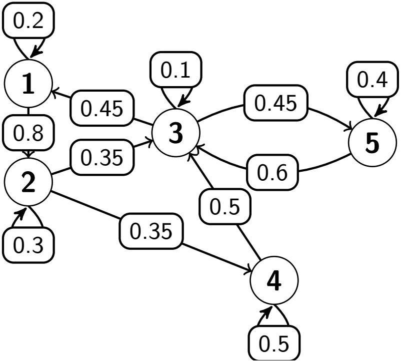
Results

- Redefining the SVO equations, it is possible to compute OAT sensitivities.
- For linear systems, the general sensitivity is equal to the sum of OAT sensitivities provided there is no initial state uncertainty.
- In the journal version, these methods are extended for the larger class of Linear Parameter-Varying (LPV) systems.
- Also, how to efficiently compute for uncertain LPV is presented where the optimality is lost.

Simulation Setup

Setup: 5 vehicles with unicycle dynamics in a formation governed by a graph.

- Vehicles start within a square of side 2 centered at the origin.
- The vehicles move using their inputs.
- After the movement, they follow a consensus algorithm.



Simulation Results

Main question: which vehicle affects the most the final state?

- Unknown initial position but known orientation;
- If one actuates node 3, it obtains the largest variation of worst-case trajectories;

# vehicle	$\mathcal{S}(X_i(H), 1)$
1	2.0467
2	2.0654
3	2.1017
4	2.0428
5	2.0763

Simulation Results

Main question: which vehicle affects the most the final state?

- With known initial position and orientation;
- The sum of OAT sensitivities are equal to the general sensitivity of 2.3329 (size of the input plus sensitivities).

# vehicle	$\mathcal{S}(X_i(H), 1)$
1	0.0467
2	0.0654
3	0.1017
4	0.0428
5	0.0763

Concluding Remarks

Contributions:

- We have presented a reachability-based method for sensitivity analysis.
- It is shown that for LTI systems, the general sensitivity is equivalent to the sum of OAT sensitivities if the initial state is known.
- In this paper, the model was assumed LTI but the SVOs work for the broader class of uncertain LPV systems (journal version).
- The more challenging case of uncertain parameters is studied in the journal version.

References

References

A. Saltelli, P. Annoni, I. Azzini, F. Campolongo, M. Ratto, and S. Tarantola, "Variance based sensitivity analysis of model output. design and estimator for the total sensitivity index," *Computer Physics Communications*, vol. 181, no. 2, pp. 259 –270, 2010, ISSN: 0010-4655. DOI: 10.1016/j.cpc.2009.09.018.

D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Stochastic and deterministic fault detection for randomized gossip algorithms," *Automatica*, vol. 78, pp. 46 –60, 2017, ISSN: 0005-1098. DOI: http://doi.org/10.1016/j.automatica.2016.12.011.

D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Fault detection for LPV systems using set-valued observers: A coprime factorization approach," *Systems & Control Letters*, vol. 106, pp. 32 –39, 2017, ISSN: 0167-6911. DOI: https://doi.org/10.1016/j.sysconle.2017.05.007.

D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Set-based fault detection and isolation for detectable linear parameter-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 18, pp. 4381–4397, 2017, rnc.3814, ISSN: 1099-1239. DOI: 10.1002/rnc.3814.

D. Silvestre, P. Rosa, J. P. Hespanha, and C. Silvestre, "Self-triggered and event-triggered set-valued observers," *Information Sciences*, vol. 426, pp. 61 –86, 2018, ISSN: 0020-0255. DOI: https://doi.org/10.1016/j.ins.2017.10.029.

The end

- Thank you for your time.

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