Source Localization and Network Topology Discovery in Infection Networks

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Outline

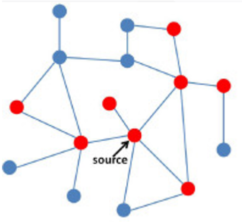
1. Introduction
2. Network Modeling
3. Source Localization
4. Network Structure Discovery
5. Simulation Results
6. Final Remarks

Background and Motivation

- Network Topology - The links or connections among the various components in a network.
- Current research trends relating to this problem include source localization and structure discovery.

Objective

- Source Localization - A general Susceptible-Infected (SI) Model used to solve the source localization problem.
- Network Topology Discovery - A new method to find the network topology with certain constraints.



SI Model

Diffusion Model

Theorem (Diffusion Model)

The SI infection model is equivalent to the dynamical system modelled by the equations:

$$x(t) = \Phi(t,0)x(0)$$
$$y(t) = Cx(t)$$

where $\Phi(t,0) = \overline{A^t + A^{t-1}}$.

Problem

The initial infection time is known and set to be the origin of the time index t. We want a general model without initial time information.

General Diffusion Model

Theorem (General Diffusion Model)

The SI infection model without knowledge of the initial infection is equivalent to the dynamical system defined by the following equations:

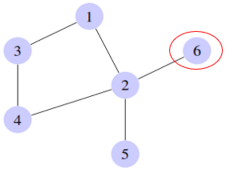
$$z(t) = \Phi(t,0)z(0) + \sum_{\tau=1}^{t} \Phi(t,\tau)u(\tau)$$
$$y(t) = Cz(t)$$

which can be rewritten as:

$$z(t) = \sum_{\tau=1}^{t} \Phi(t-\tau,0)u(\tau)$$
$$y(t) = Cz(t)$$

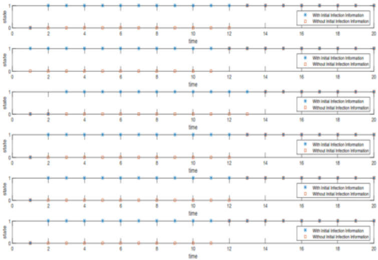
Example

- The infection time is assumed to be t=10 at node six when the source impulse is added to the network.
- The observers are set to be node one and three and the total running time is set to be 20 time instants.



Example Simulation Results

- Referring to Theorem 2 [General Diffusion Model], the general state equations should produce the same state evolution shifted by ten time units.



Observability

- The source localization problem can be viewed as the state estimation of a dynamical model, one needs to discuss the observability of the system;
-

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix} \quad \begin{bmatrix} C \\ C\Phi(1,0) \\ \vdots \\ C\Phi(N,0) \end{bmatrix} \quad x(0) \tag{1}$$

or equivalently,

$$Y_N = O_N x(0) \tag{2}$$

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Observability

Theorem (Observability)

If the rank of the observability matrix O_N is equal to n, for the particular choice of observers, the initial state can be obtained by

$$x(0) = (O_N^T O_N)^{-1} O_N^T Y_N \tag{3}$$

- This Theorem only contains the network structure and the location of the output nodes. The source localization does not depend on the initial infection time.

Source Localization With Initial State Information

- We assume the initial state of the network is $\exists 1 \leq i \leq n : x(0) = e_i$, where e_i is the canonical vector.
- The equation $x(0) = (O_N^T O_N)^{-1} O_N^T Y_N$ in Theorem 3 [Observality] can be rewritten as a rank minimization problem with constraints dependent on the measurements performed by the observers.
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$$\min_{x(0)} \ ||x(0)||_0 \tag{4}$$

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l_1 minimization

- The previous l_0 pseudo norm optimization attains its minimum for the sparsest $x(0)$ that satisfy the constraints.
- Since l_0 is not a norm, the optimization becomes non-convex, resulting in an NP hard problem.
- If the system is non-observable, an approximation to the solution of the linear equation can be found by resorting to the l_1 minimization problem

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Source Localization Without Initial State Information

- Introduce a variable size matrix $O(\tau)$ that corresponds to observability matrix for the first $t-\tau$ observations of the general model as

$$O(\tau) = \begin{bmatrix} C \\ C\Phi(1,0) \\ \vdots \\ C\Phi(t-\tau-1,0) \end{bmatrix} \quad (6)$$

- from Theorem 2 [GeneralModel],we have $z(t) = \sum_{\tau=1}^{t} \Phi(t,\tau)u(\tau)$, which is also determined by τ for each time instant.

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How to solve unknown intial time

- $x(t) = \Phi(t, 0)x(0)$
- Determined by t.

- $z(t) = \sum_{\tau=1}^{t} \Phi(t, \tau)u(\tau)$
- Determined by t and τ.

- The matrix size of $O(\tau)$ will be determined and we can write $Y_{N+\tau} = O_{N+\tau}(\tau)u(\tau)$ and it can be obtained by l_1 norm optimization as

$$\min_{u(\tau)} \; ||u(\tau)||_1 \tag{7}$$

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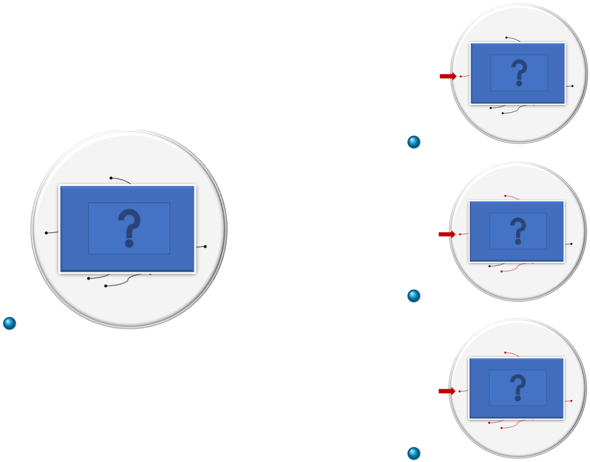
Principle



Main Steps

- $$Y_{N+\tau} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N+\tau) \end{bmatrix} = \begin{bmatrix} Cx(0) \\ Cx(1) \\ \vdots \\ Cx(N+\tau) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N+\tau) \end{bmatrix} \tag{8}$$

- $$P(i,j) = P(j,i), \forall 1 \leq i \leq n$$

- $$P(i,i) = 0, \forall 1 \leq i \leq n$$

- $$x_{cal} = \begin{cases} (P^t + P^{t-1})x(0) \geq 1 & x(t) = 1 \\ (P^t + P^{t-1})x(0) \leq 0 & x(t) = 0 \end{cases}$$

Other Solvers

- The network structure can also be found resorting to satisfiability solvers, as a Boolean satisfiability problem, which is a NP-complete problem
- SAT Competition 2016 reflects recent developments with certain experimental results, which contains statistical data showing SAT-VERIFIED or UNSAT and only returns one possible result once it is satifiable, which means that the normal SAT solver has no guarantee of uniqueness.

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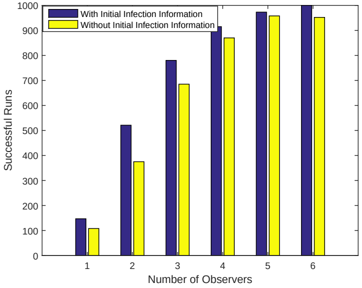
Simulation Results (Source Localization)[1/3]

To distinguish the difference between the model with initial state information and the general model without the intial state information in practice.

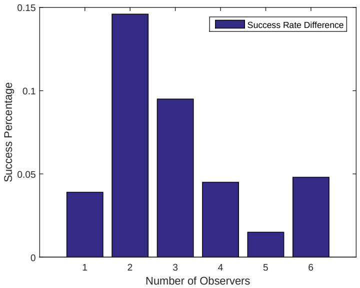
- Similarly, with the increase of observer numbers, there are more successful runs.
- Always less successful runs of the general model than the successful runs of the model with initial state information.
- This is the infection time mischosen, which is due to the occasional mistaken source localization with a $solve - avaliable$ timeshifting.



Simulation Results (Source Localization)[2/3]

Compare the difference bewteen two models in last page.

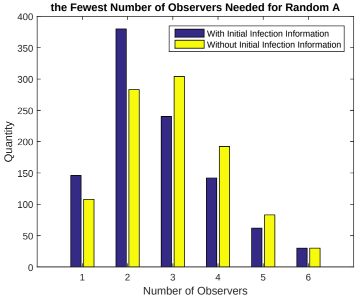
- The successful percent difference is discreasing almost linely from about 0.14 to 0.02 for observers number 2 to 5, while the difference of observer number 1 and observers number 6 are almost same.
- The reason of the successful percent difference becoming smaller with the increase of observer number from 2 to 5 is that with more constraints provided by obersvers, it is more likely that the optimization process will localize the source correctly. Also, the increase of the constraints effect the accuracy of the genreal model more than the model with the initial state information, which makes the successful percent difference smaller. However, when the number of observers is 1 or 6, the system is almost not observed or totally observed. Thus, the network structure can be seldomly found or almostly found, which cause the successful percent difference close to a constant 0.04.



Simulation Results (Source Localization)[3/3]

Setup: Fix the structure of a network, and then increase the number of observers from 1 to 6 to discover how many observers is indeed needed to find its source localization and summarize the most frequent number of observers needed for a random network structure.

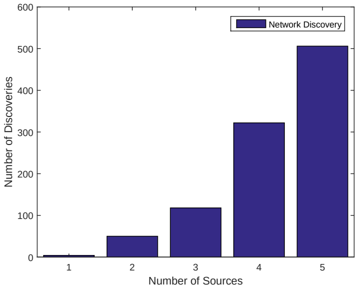
- The fewest number of observers needed for a random network structure is different for two models, which is same as what we have predicted due to $solve - avaliable$ timeshifting problem.



Simulation Results (Network Structure Discovery)

Setup: An unknown random network structure is constructed and a single source is *injected* to observe it. Once a single source failed, another single source is injected to discover the network structure.

- For one thousand random network structures, about half of them need to have 5 different sources being *injected* into the network.
- The number of sources for network discovery is five-ladder like distribution, which indicates the constraints for a totally unknown network structure discovery is needed to be fully enough.



Final Remarks

Contributions:

- An extension to the diffusion model to allow for an unknown infection time;
- A linear program based network topology discovery algorithm for the diffusion model.

The end

- Thank you for your time.

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