Dark Radiation and Localization of Gravity on the Brane

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Abstract

We discuss the dynamics of a spherically symmetric dark radiation vaccum in the Randall-Sundrum brane world scenario. Under certain natural assumptions we show that the Einstein equations on the brane form a closed system. For a de Sitter brane we determine exact dynamical and inhomogeneous solutions which depend on the brane cosmological constant, on the dark radiation tidal charge and on its initial configuration. We define the conditions leading to singular or globally regular solutions. We also analyse the localization of gravity near the brane and show that a phase transition to a regime where gravity propagates away from the brane may occur at short distances during the collapse of positive dark energy density.

1 Introduction

In the search for extra spatial dimensions the Randall and Sundrum (RS) brane world scenario is particularly interesting for its simplicity and depth [1]. In this model the Universe is a 3-brane boundary of a noncompact Z_2 symmetric 5-dimensional anti-de Sitter space. The matter fields live only on the brane but gravity inhabits the whole bulk and is localized near the brane by the warp of the infinite fifth dimension.

Since its discovery many studies have been done within the RS scenario (see Ref. [2] for a recent review and notation). For a brane bound observer [3, 4, 5] the interaction between the brane and the bulk introduces correction terms to the 4-dimensional Einstein equations, namely, a local high energy embedding term generated by the matter energy-momentum tensor and a non-local term induced by the bulk Weyl tensor. Such equations have an intrincate non-linear dynamics. For example, the exterior vaccum of collapsing matter on the brane is now filled with gravitational

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modes originated by the bulk Weyl curvature and can no longer be regarded as a static space [6, 7].

Previous research on the RS scenario has been focused on static or homogeneous dynamical solutions. In this proceedings we report some new results on the dynamics of a spherically symmetric RS brane world vaccum. For a de Sitter brane we present exact dynamical and inhomogeneous solutions, define the conditions to characterize them as singular or globally regular and discuss the localization of gravity to the vicinity of the brane (see Ref. [8] for more details).

2 Brane Vaccum Field Equations

In the Gauss-Codazzi formulation of the RS model [3, 4, 5], the Einstein vaccum field equations on the brane are given by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{1}$$

where Λ is the brane cosmological constant and the tensor $\mathcal{E}_{\mu\nu}$ is the limit on the brane of the projected 5-dimensional Weyl tensor. It is a symmetric and traceless tensor constrained by the following conservation equations

$$\nabla_\mu \mathcal{E}^\mu_\nu = 0. \tag{2}$$

The projected Weyl tensor $\mathcal{E}_{\mu\nu}$ can be written in the following general form [5]

$$\mathcal{E}_{\mu\nu} = -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \left[\mathcal{U}\left(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}\right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu\right], \tag{3}$$

where u_μ such that $u^\mu u_\mu = -1$ is the 4-velocity field and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the tensor projecting orthogonaly to u_μ. The forms $\mathcal{U}$, $\mathcal{P}_{\mu\nu}$ and $\mathcal{Q}_\mu$ represent different aspects of the effects induced on the brane by the 5-dimensional gravitational field. Thus, $\mathcal{U}$ is an energy density, $\mathcal{P}_{\mu\nu}$ a stress tensor and $\mathcal{Q}_\mu$ an energy flux.

Since the 5-dimensional metric is not known, in general $\mathcal{E}_{\mu\nu}$ is not completely determined on the brane [3, 4] and so the effective 4-dimensional theory is not closed. To close it we need simplifying assumptions about the effects of the gravitational field on the brane. For instance we may consider a static and spherically symmetric brane vaccum with $\mathcal{Q}_\mu = 0$, $\mathcal{P}_{\mu\nu} \neq 0$ and $\mathcal{U} \neq 0$. This leads to the Reissner-Nordström black hole solution on the brane [9].

It is also possible to close the system of Einstein equations when considering a dynamical and spherically symmetric brane vaccum with $\mathcal{Q}_\mu = 0, \mathcal{U} \neq 0$, and $\mathcal{P}_{\mu\nu} \neq 0$. The general, spherically symmetric metric in comoving coordinates (t, r, θ, ϕ) is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^\sigma dt^2 + A^2 dr^2 + R^2 d\Omega^2, \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $\sigma = \sigma(t,r)$, $A = A(t,r)$, $R = R(t,r)$ and R is the physical spacetime radius. If the traceless stress tensor $\mathcal{P}_{\mu\nu}$ is isotropic then it will have the general form

$$\mathcal{P}_{\mu\nu} = \mathcal{P}\left(r_\mu r_\nu - \frac{1}{3}h_{\mu\nu}\right), \tag{5}$$

where $\mathcal{P} = \mathcal{P}(t,r)$ and r_μ is the unit radial vector, given in the above metric by $r_\mu = (0, A, 0, 0)$. Then

$$\mathcal{E}_\mu^\nu = \left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \mathrm{diag}\left(\rho, -p_r, -p_T, -p_T\right), \tag{6}$$

where the energy density and pressures are, respectively, $\rho = \mathcal{U}$, $p_r = (1/3)\left(\mathcal{U} + 2\mathcal{P}\right)$ and $p_T = (1/3)\left(\mathcal{U} - \mathcal{P}\right)$. Consequently, the conservation Eq. (2) read [10]

$$\begin{aligned} 2\frac{\dot{A}}{A}\left(\rho + p_r\right) &= -2\dot{\rho} - 4\frac{\dot{R}}{R}\left(\rho + p_T\right), \\ \sigma'\left(\rho + p_r\right) &= -2{p_r}' + 4\frac{R'}{R}(p_T - p_r), \end{aligned} \tag{7}$$

where the dot and the prime denote, respectively, derivatives with respect to t and r. A synchronous solution is permitted with the equation of state $\rho + p_r = 0$, equivalent to $\mathcal{P} = -2\mathcal{U}$ where $\mathcal{U}$ has the dark radiation form

$$\mathcal{U} = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 \frac{Q}{R^4}. \tag{8}$$

The constant Q is the dark radiation tidal charge. Hence, we get

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \frac{Q}{R^4}\left(u_\mu u_\nu - 2r_\mu r_\nu + h_{\mu\nu}\right), \tag{9}$$

an exactly solvable closed system for the unknown functions $A(t,r)$ and $R(t,r)$ which depends on the free parameters Λ and Q. Indeed, its solutions are of the LeMaître-Tolman-Bondi type

$$ds^2 = -dt^2 + \frac{{R'}^2}{1+f}dr^2 + R^2 d\Omega^2, \tag{10}$$

where R satisfies

$$\dot{R}^2 = \frac{\Lambda}{3}R^2 - \frac{Q}{R^2} + f. \tag{11}$$

The function $f = f(r) > -1$ is interpreted as the energy inside a shell labelled by r in the dark radiation vaccum and is fixed by its initial configuration.

3 Localization of Gravity near the Brane

As is clear in Eq. (9) the dark radiation dynamics depends on Λ and Q. It is important to point out that these parameters have a direct effect on the localization of gravity in the vicinity of the brane. Indeed, the tidal acceleration away from the brane [5] is given by [9]

$$
-\lim_{y\to 0\pm}\tilde{R}_{ABCD}n^A\tilde{u}^B n^C\tilde{u}^D = \frac{\tilde{\kappa}^2}{6}\tilde{\Lambda} + \frac{Q}{R^4}, \tag{12}
$$

where $\tilde{u}_A$ is the extension off the brane of the 4-velocity field satisfying $\tilde{u}^A n_A = 0$ and $\tilde{u}^A\tilde{u}_A = -1$. The gravitational field is only bound to the brane if the tidal acceleration points towards the brane. It must then be negative implying that

$$
\tilde{\Lambda}R^4 < -\frac{6Q}{\tilde{\kappa}^2}. \tag{13}
$$

As a consequence, gravity is only localized for all R if $\Lambda < \Lambda_c$ with $\Lambda_c = \tilde{\kappa}^4\lambda^2/12$ and $Q \leq 0$ or $\Lambda = \Lambda_c$ and $Q < 0$. For $\Lambda < \Lambda_c$ and $Q > 0$ the gravitational field will just remain localized if $R > R_c$ where $R_c^4 = 3Q/(\Lambda_c - \Lambda)$. On the other hand for $\Lambda > \Lambda_c$ and $Q < 0$ localization is limitted to the epochs $R < R_c$. If $\Lambda \geq \Lambda_c$ and $Q \geq 0$ then gravity is always free to propagate far away into the bulk.

According to recent supernovae measurements (see *e.g.* Ref. [11]) $\Lambda \sim 10^{-84}\mathrm{GeV}^2$. On the other hand $\tilde{M}_{\mathrm{p}} > 10^8\mathrm{GeV}$ and $M_{\mathrm{p}} \sim 10^{19}\mathrm{GeV}$ imply $\lambda > 10^8\mathrm{GeV}^4$ [12] because $6\kappa^2 = \lambda\tilde{\kappa}^4$. Since $\Lambda_c = \kappa^2\lambda/2$ then Λ_c is bound from below, $\Lambda_c > 10^{-29}\mathrm{GeV}^2$. Hence, observations demand Λ to be positive and smaller than the critical value Λ_c, $0 < \Lambda < \Lambda_c$. Note that is means an anti-de Sitter bulk, $\tilde{\Lambda} < 0$. The same conclusion is true if $\tilde{M}_{\mathrm{p}}$ is in the TeV range because Λ_c increases when $\tilde{M}_{\mathrm{p}}$ decreases.

Since current observations do not yet constrain the sign of Q [13] we conclude that for $0 < \Lambda < \Lambda_c$ only for $Q < 0$ gravity is bound to the brane for all R. If $Q > 0$ then for $R < R_c$ the tidal acceleration is positive and gravity is no longer localized near the brane.

4 Inhomogeneous Dynamics for a de Sitter Brane

Assume from now on that $0 < \Lambda < \Lambda_c$. Non-static solutions correspond to $f \neq 0$. An example is

$$
\left|R^2 + \frac{3f}{2\Lambda}\right| = \sqrt{\beta}\cosh\left[\pm 2\sqrt{\frac{\Lambda}{3}}t + \cosh^{-1}\left(\frac{\left|r^2 + \frac{3f}{2\Lambda}\right|}{\sqrt{\beta}}\right)\right], \tag{14}
$$

where $\beta = (3/\Lambda)[3f^2/(4\Lambda) + Q]$ and $+$ or $-$ correspond respectively to expansion or collapse. If $Q > 0$ then $f > -1$ but for $Q < 0$ the energy function f must satisfy in addition $|f| > 2\sqrt{-Q\Lambda/3}$. Since R is a non-factorizable function of t and r

these solutions define new exact and inhomogeneous cosmologies for the spherically symmetric dark radiation de Sitter brane.

5 Singularities and Regular Bounces

The dark radiation dynamics defined by Eq. (11) may produce shell focusing singularities at $R = 0$ or regular bouncing points at some $R \neq 0$. To see this consider

$$R^2 \dot{R}^2 = V(R,r) = \frac{\Lambda}{3} R^4 + f R^2 - Q. \tag{15}$$

If for all $R \geq 0$ the potential V is positive then a shell focusing singularity forms at $R = R_s = 0$. Alternatively, if there is an epoch $R = R_* \neq 0$ for which $V = 0$ then a regular rebounce point appears at $R = R_*$. For the dark radiation vaccum at most two regular rebounce epochs can be found. Since $\Lambda > 0$ there is always a phase of continuous expansion to infinity with ever increasing rate. Depending on $f(r)$ other phases may exist. To ilustrate take $\beta > 0$ and compare the settings $Q < 0, f > -1, |f| > 2\sqrt{-Q\Lambda/3}$ and $Q > 0, f > -1$.

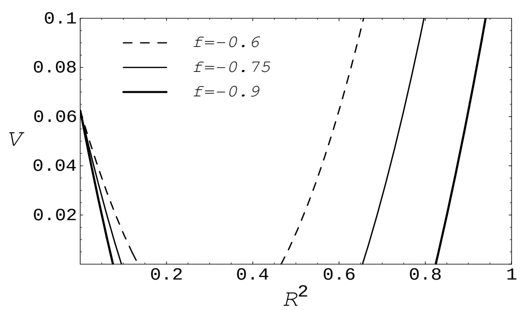

Figure 1: Plot of V for $\beta > 0$ and $Q < 0$. Non-zero values of f belong to the interval $-1 < f < -2\sqrt{-Q\Lambda/3}$ and correspond to shells of constant r.

If for $Q < 0$ we have $f > 2\sqrt{-Q\Lambda/3}$ then $V > 0$ for all $R \geq 0$. There are no rebounce points and the dark radiation shells may either expand continuously or collapse to a singularity at $R_s = 0$. However for $-1 < f < -2\sqrt{-Q\Lambda/3}$ (see Fig. 1) we find two rebounce epochs at $R = R_{*\pm}$ with $R^2_{*\pm} = -3f/(2\Lambda) \pm \sqrt{\beta}$. Since $V(0,r) = -Q > 0$ a singularity also forms at $R_s = 0$. Between the two rebounce points there is a forbidden zone where V is negative. The phase space of allowed dynamics is thus divided in two disconnected regions separated by the forbidden interval $R_{*-} < R < R_{*+}$. For $0 \leq R \leq R_{*-}$ the dark radiation shells may expand to a maximum radius $R = R_{*-}$, rebounce and then fall to the singularity. If $R \geq R_{*+}$ then there is a collapsing phase to the minimum radius $R = R_{*+}$ followed by reversal and subsequent accelerated continuous expansion. The singularity at $R_s = 0$ does not form and so the solutions are globally regular. Since $Q < 0$ gravity is bound to the brane for all the values of R.

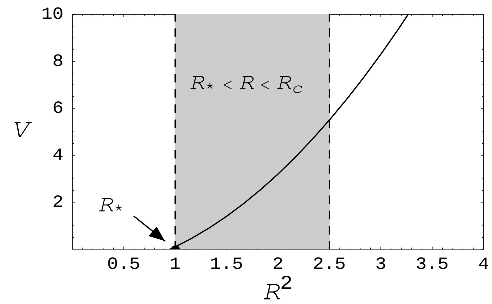

Figure 2: Plot of V for $\beta > 0$ and $Q > 0$. Non-zero values of f belong to the interval $f > -1$ and correspond to shells of constant r. The shaded region indicates where gravity is not localized near the brane.

If $Q > 0$ (see Fig. 2) then we find globally regular solutions with a single rebounce epoch at $R = R_*$ where $R_*^2 = -3f/(2\Lambda) + \sqrt{\beta}$. This is the minimum possible radius for a collapsing dark radiation shell. It then reverses its motion and expands forever. The phase space of allowed dynamics defined by V and R is limitted to the region $R \geq R_*$. Below R_* we find a forbidden region where V is negative. In particular, $V(0, r) = -Q < 0$ implying that the singularity at $R_s = 0$ does not form and so the solutions are globally regular. Note that if gravity is to be bound to the the brane for $R > R_*$ then $R_* > R_c$. If not then we find a phase transition epoch $R = R_c$ such that for $R \leq R_c$ the gravitational field is no longer localized near the brane.

6 Conclusions

In this work we have reported some new results on the dynamics of a RS brane world dark radiation vaccum. Using an effective 4-dimensional approach we have shown that some simplifying but natural assumptions lead to a closed and solvable system of Einstein field equations on the brane. We have presented a set of exact dynamical and inhomogeneous solutions for $\Lambda > 0$ showing they further depend on the dark radiation tidal charge Q and on the energy function $f(r)$. We have also described the conditions under which a singularity or a regular rebounce point develop inside the dark radiation vaccum and discussed the localization of gravity near the brane. In particular, we have shown that a phase transition to a regime where gravity is not bound to the brane may occur at short distances during the collapse of positive dark energy density on a realistic de Sitter brane. Left for future research is for example an analysis of the dark radiation vaccum dynamics from a 5-dimensional perspective.

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